

Gravity *à la* Born-Infeld

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Abstract

A simple technique for the construction of gravity theories in Born-Infeld style is presented, and the properties of some of these novel theories are investigated. They regularize the positive energy Schwarzschild singularity, and a large class of models allows for the cancellation of ghosts. The possible correspondence to low energy string theory is discussed. By including curvature corrections to all orders in α' , the new theories nicely illustrate a mechanism that string theory might use to regularize gravitational singularities.

PACS numbers: 04.90.+e, 04.20.Cv, 04.50.+h

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I. INTRODUCTION

The problem of the diverging Coulomb field and self-energy of point particles in Maxwell's theory of electrodynamics led Born and Infeld in the 1930s to the construction of a non-linear extension of this theory [1] in which a regularization is achieved because the field strength tensor F_{ab} appears to all orders. A particular appeal of Born-Infeld electrodynamics is that its action can be expressed in the compact determinantal form

$$\int \sqrt{-\det(\eta_{ab} + \lambda F_{ab})} \quad (1)$$

and, more importantly, it stems from the fact that the Dirac-Born-Infeld action resurfaces in string theory as the low energy effective action [2] on the worldvolume of D -branes, by capturing the degrees of freedom of open strings in an abelian background gauge field. The Born-Infeld parameter λ is then related to the inverse tension of fundamental strings, by $\lambda = 2\pi\alpha'$.

Einstein's general relativity is the fundamental building block for the gravity sector of superstring or M-theory which consistently unify gravity and the standard model of elementary particle physics. In supersymmetrized form, general relativity appears as the low energy effective theory (in ten or eleven dimensions). That it is plagued, like Maxwell electrodynamics, by singularities in the gravitational field of point masses, thus can be considered a major problem. Of course, string/M-theory predict corrections to general relativity at higher orders in the curvature (corresponding to higher orders in α'), but no way has been found yet to resum these corrections, as in the Born-Infeld case, so to incorporate arbitrary high powers of the curvature and to solve the problems inherent in Einstein gravity.

Inspired by Born-Infeld electrodynamics, several attempts have been made in the construction of gravity analogues. Deser and Gibbons [3] formulate three obvious criteria that such a theory should satisfy.

1. Freedom of ghosts.
2. Regularization of singularities.
3. Supersymmetrizability.

Some comments on these points are in order.

ad 1. Responsible for the propagation of ghosts, in a perturbative expansion around the Minkowski vacuum spacetime, are the terms in the action which are quadratic in the curvature. The only exemption from this rule is the quadratic Gauss-Bonnet combination

$$\sqrt{-g}(R^2 - 4R_{ab}^2 + R_{abcd}^2) \quad (2)$$

which is a total derivative in four dimensions and for which, in higher dimensions, the quadratic terms in the perturbative expansion still cancel up to total derivatives [4].

ad 2. In general, not all types of singularities can be removed, and not all of them should be removed, as pointed out by Horowitz and Myers [5]. They argue that certain gravitational plane waves are solutions to any general theory whose Lagrangian is a function of the metric, the curvature and its covariant derivatives. This is due to the fact that all curvature scalars of these spacetimes vanish. But the two polarizations of a plane wave can diverge, which leads to unbounded tidal forces. Hence, all extensions of general relativity will have solutions with null singularities. Furthermore, negative energy solutions of general relativity, e.g., the Schwarzschild solution with negative mass parameter, should not be regularized, since this would imply an instability of the Minkowski vacuum. The singularities of negative energy solutions serve as a useful criterion of unphysicality.

ad 3. Supersymmetrizability is a very stringent requirement and is presumably natural if the gravity theory were implied by higher-dimensional string/M-theory.

It is illuminating to take a closer look at previous attempts to construct gravities in Born-Infeld style. The discussion of [3], for instance, considers a Lagrangian of the form $(-\det(c_1 g_{ab} + c_2 R_{ab} + c_3 X_{ab}))^{1/2}$ where the tensor X_{ab} contains terms of second or higher order in the curvature and the c_i are arbitrary constants. The simplest of these models has $c_1 = 1$ and $c_3 = 0$. It is, of course, not ghost-free because terms from X_{ab} are needed to balance those arising in second order of the curvature expansion from R_{ab} . This model does not improve general relativity in the sense that it has the standard Schwarzschild (-de Sitter) solution. Note that this can be explained intuitively. The Schwarzschild solution is Ricci flat, and hence, its singularity can be recognized only from the second order curvature invariant R_{abcd}^2 , but not from R^2 and R_{ab}^2 . But this invariant does not occur in the curvature expansion of the above model, so that it may be unbounded in solutions. A theory that places an upper bound on R_{abcd}^2 is introduced in a rather *ad hoc* way in [6], essentially by adding $c_1(1 - c_2 R_{abcd}^2)^{1/2}$ to the Einstein-Hilbert Lagrangian. The author of this paper finds

non-singular black hole solutions, which underlines the importance of using the full Riemann tensor in order to regularize gravitational singularities. Supersymmetric constructions, which will not be further discussed here, can be found in [7].

Making use of these insights, in this article, a method is presented to construct a large, and novel, class of gravity theories in Born-Infeld style. Section II introduces the basic model and discusses some of its properties. In particular, it is found that the positive mass Schwarzschild singularity may be removed, whereas the negative energy solution remains singular as is necessary for a stable Minkowski vacuum. Section III considers generalizations of the basic model that allow for the cancellation of ghosts. These generalized models are compared and matched to the low energy effective gravity action following from string theory, in section IV. The article concludes with a discussion in section V.

II. THE BASIC MODEL

A. Action and equations of motion

Start out with the following two basic requirements. In order to be as close in spirit as possible to the structure of Born-Infeld electrodynamics, try to find a gravity action, on a d -dimensional pseudo-Riemannian manifold with metric g_{ab} , which can be written in a nice determinantal form, and incorporate the full Riemann tensor in order to enable a possible regularization of the singularities. Both requirements can be satisfied by using the symmetries of the Riemann tensor calculated from the metric connection. It is antisymmetric in two pairs of indices, i.e., it satisfies $R_{abcd} = R_{[ab]cd} = R_{ab[cd]}$. Introduce the symmetric tensor

$$R_{AB} \equiv R_{[a_1 a_2][b_1 b_2]} \quad (3)$$

whose capital indices take $d(d-1)/2$ values that can be imagined as ordered pairs of standard indices. Along with this definition, introduce a new metric and Kronecker delta obeying the same symmetries,

$$g_{AB} \equiv g_{a_1 b_1} g_{a_2 b_2} - g_{a_2 b_1} g_{a_1 b_2} , \quad (4a)$$

$$\delta_B^A \equiv \delta_{b_1}^{a_1} \delta_{b_2}^{a_2} - \delta_{b_1}^{a_2} \delta_{b_2}^{a_1} . \quad (4b)$$

These tensors can be used to upper and lower, or replace, capital indices as usual. For the metric, a useful determinant formula holds,

$$\det g_{AB} = (\det g_{ab})^{d-1}. \quad (5)$$

This suggests the following basic model of a Born-Infeld type gravity, valid in an arbitrary dimension d . Consider the action

$$\int (-\det(g_{AB} + \lambda R_{AB}))^\zeta = \int \sqrt{-g} (\det(\delta_B^A + \lambda R_B^A))^\zeta \quad (6)$$

with a parameter λ of the dimension length squared. The equality holds for an exponent $\zeta = \frac{1}{2(d-1)}$. Note, however, that one might start out from the action on the right hand side which allows for different ζ . The only restriction appears to be a fractional ζ because a possible regularization of singularities is only expected from the appearance of all curvature orders in the action. A curvature expansion of the Lagrangian is equivalent to an expansion around $\lambda = 0$ and, being a special case of equation (A9) with $M_B^A = R_B^A$ and $N_B^A = 0$, gives

$$1 + \frac{1}{2}\zeta\lambda \left(R + \frac{1}{4}\zeta\lambda R^2 - \frac{1}{4}\lambda R_{abcd}^2 \right) + \mathcal{O}(\lambda^3) \quad (7)$$

for the lowest orders. This shows that it is necessary to subtract the cosmological constant that is implicit above, in order to obtain the Einstein-Hilbert Lagrangian in the limit $\lambda \rightarrow 0$. The amended action considered in the following is

$$\int \sqrt{-g} \left[(\det(\delta_B^A + \lambda R_B^A))^\zeta - 1 \right]. \quad (8)$$

For small λ this theory inherits all experimental tests of general relativity. The curvature expansion, however, also shows that a quantization of this classical theory would include ghost modes because the quadratic curvature terms do not appear in the Gauss-Bonnet combination; see (2) in the introduction. The cancellation of the ghosts will be discussed in the following section III.

For the moment, it is useful to push on and to investigate some properties of this basic model which are carried over, at least qualitatively, to the more complicated models. The equations of motion are derived by varying the action (8) with respect to the standard spacetime metric g_{ab} . They read

$$\begin{aligned} & (\det(\delta + \lambda R))^\zeta \left[g^{cd} + \frac{1}{2}\zeta\lambda (\delta + \lambda R)^{-1}{}^B{}_{a_1 a_2} \left(g^{c[a_1} R^{a_2]d}{}_B + g^{d[a_1} R^{a_2]c}{}_B \right) \right] - g^{cd} \\ & + \frac{1}{2}\zeta\lambda \delta_{[b_1}^c g^{d[a_1} (\nabla^{a_2]} \nabla_{b_2]} + \nabla_{b_2]} \nabla^{a_2]} \left[(\det(\delta + \lambda R))^\zeta (\delta + \lambda R)^{-1}{}^{b_1 b_2}{}_{a_1 a_2} \right] = 0. \end{aligned} \quad (9)$$

Terms in $\mathcal{O}(1)$ cancel in the limit $\lambda \rightarrow 0$. This is enforced by the subtraction of the cosmological constant in the action, which is responsible for the subtraction of the term $-g^{cd}$ in the equations of motion. The second line is of $\mathcal{O}(\lambda^2)$ and vanishes. So, in $\mathcal{O}(\lambda)$, Einstein's vacuum equations $R^{cd} - \frac{1}{2}Rg^{cd} = 0$ remain from the first line, as they should.

B. Schwarzschild without singularities

Now consider the analogue of the spherically symmetric Schwarzschild solution of general relativity. It turns out to be easier, than using the equations of motion given above, to derive an effective action for the metric ansatz

$$g_{ab} = \text{diag}(-A(r), B(r), r^2, r^2 \sin^2 \theta) \quad (10)$$

where the standard Schwarzschild coordinates are given by $\{t, r, \theta, \phi\}$. Such an approach usually is only valid in very special situations, where it can be shown that the symmetry reduction by means of a specific ansatz and the variational principle, used to derive the equations of motion, commute [8]. This is true for the spherically symmetric ansatz and has been widely used in the literature, e.g., in [9, 10]. The effective action is obtained to be

$$\int dr r AB \left[\frac{1}{2} A^{-7/6} B^{-2} ((\lambda + B(r^2 - \lambda))(2rAB + \lambda A')^2 (2rB^2 - \lambda B')^2 \right. \\ \left. \times (4A^2 B^2 - \lambda(AA'B' + A'^2 B - 2AA''B)))^{1/6} - 1 \right] \quad (11)$$

in four dimensions and with the exponent $\zeta = \frac{1}{2(d-1)}$. (The results of this section are expected to hold qualitatively for higher d and different fractional exponent ζ as well. Five dimensions and, for ζ , various fractional values between zero and one have been checked numerically.)

Inspired by the general relativity solution, simplify by setting $B = 1/A$. Now again, it is not immediately clear whether the effective action approach stays valid under this additional assumption, and it is also not obvious which symmetry might be responsible for that. The specific solutions which will be derived below, however, can be checked to satisfy also the separate equations of motion for A and B following from the above action. So here, $B = 1/A$ is admissible (this is, for instance, not the case for the Einstein-Hilbert action). The term associated to the subtraction of the minus one becomes independent of the function A and can be dropped. Furthermore, the action can be rewritten compactly in terms of an auxiliary function

$$D(r) = r^2 + \lambda(A(r) - 1) \quad (12)$$

as

$$\int dr r (DD'^4 D'')^{1/6}. \quad (13)$$

The equation of motion, finally, becomes a fourth order ordinary differential equation for the function $D(r)$,

$$\begin{aligned} & 5rD'^4 D''^2 - 2DD'^2 D'' [13rD''^2 + D'(6D'' - 5rD''')] + D^2 [-40rD''^4 \\ & + 8D'D''^2(12D'' + 5rD''') + 5D'^2(-11rD''^2 + 6D''(2D''' + rD'''))] = 0. \end{aligned} \quad (14)$$

The ‘trivial’ solutions of this equation, analytic in r around $r = 0$, are $c_1 r^3$, $c_1 r^2$ and $c_1 r + c_2$ for arbitrary constants c_i . Of these solutions, $D(r) = r^2$ gives Minkowski space where $A(r) = 1$.

In order to calculate the spherically symmetric gravitational field, modified with respect to general relativity, and to obtain Minkowski space asymptotically, the boundary condition $A(r) \rightarrow 1$ as $r \rightarrow \infty$ has to be satisfied. This suggests a solution for $D(r)$ in terms of a power series, essentially in the inverse radius r^{-1} ,

$$D(r) = r^2 \left(1 + \sum_{n=1}^{\infty} a_{3n} r^{-3n} \right), \quad (15)$$

where it turns out that only every third term contributes. A substitution of this expansion into the equation of motion allows solving sequentially for the unknown coefficients a_{3n} that are found to have the form $a_{3n} = b_{3n} a_3^n$ for $n > 1$ and a set of positive numbers b_{3n} , e.g., $b_6 = \frac{11}{94}$, $b_9 = \frac{1163}{19317} \dots$. From the Schwarzschild boundary condition, $A(r) \sim 1 - \frac{2m}{r}$ as r approaches infinity, and from the definition of $D(r)$ in (12) follows a relation of the coefficient a_3 to the Schwarzschild mass parameter m ,

$$a_3 = -2m\lambda. \quad (16)$$

Look at the analogue of the positive energy Schwarzschild solution first, i.e., set $m > 0$ and $\lambda > 0$. The convergence of the power series solution breaks down for $r^3 < r_0^3 \equiv |a_3|$. A numerical solution of the differential equation, for negative a_3 , can be obtained up to this point which turns out to be special. At r_0 , the function D vanishes, but $D'(r_0) > 0$. The equation of motion is satisfied by $D''(r_0) = 0$. The second derivative at this point, however, tends to zero with an infinite positive slope and an infinite negative curvature. So the solution for D becomes singular at r_0 . In principle, the numerical integration of the

equation of motion could be carried on to smaller values $r < r_0$, but this would necessitate an exact knowledge of the type of singularity involved.

Lacking such knowledge about the singularity of the solution at r_0 , it is still interesting to ask which solutions on the inner domain might match to the outer one. A way to integrate further inwards is to simply assume a pole in the third derivative of D and to supply new initial conditions at some value of r slightly smaller than r_0 . Another possibility is to look for solutions to the equation of motion which are analytic in r at $r = 0$ and obey the very weak condition that their function value match the outer solution at r_0 . The latter case is extremely restrictive, and there are only two possibilities. The first is the linear function $D(r) = D'(r_0)(r - r_0)$, and the second is an identically vanishing $D(r) \equiv 0$. In any case, it is impossible to smoothly match all derivatives at r_0 . The outer solution and the possible inner solutions are shown in figure 1; the numerically continued solution approximates the linear $D(r)$ to a high degree of accuracy. Note that all possible inner solutions regularize the metric function $A(r)$ at $r = 0$. The curvature singularities, however, are only regularized by the inner solution with vanishing $D(r)$. In this case, one finds $R = -\frac{6}{\lambda}$, $R_{ab}^2 = \frac{36}{\lambda^2}$ and $R_{abcd}^2 = \frac{6}{\lambda^2}$ throughout the inner domain.

The completely regularized solution is very interesting for the following reason. Vanishing $D(r) \equiv 0$ corresponds to a Schwarzschild function $A(r) = 1 - r^2/\lambda$ on the inner domain and thus gives the spherical solution a de Sitter core. This scenario is reminiscent of the ‘gravastar’ picture, see [11, 12] and compare [13]. Based on the assumption of the incompatibility of the standard Schwarzschild spacetime with quantum mechanics, these authors develop a setup in which this spacetime is replaced, in the interior, by a segment of de Sitter space, separated by a shell, filled with a ‘quantum fluid’, from the exterior Schwarzschild solution. These models are hoped to resolve among other problems the black hole information paradox. Intriguingly, the gravity model proposed here enforces the existence of a special radius at which a de Sitter core may be merged to an approximately Schwarzschild outer domain, even without relying on the *ad hoc* introduction of any quantum fluid. Admittedly, the physical interpretation of the positive energy solution near the special point r_0 has to be worked out in more detail.

In the case of the negative energy analogue of the Schwarzschild solution, the mass parameter m is negative, which makes the series coefficient a_3 positive (keeping $\lambda > 0$). Now each term of the power series solution diverges for small radii $r^3 < |a_3|$, so the whole

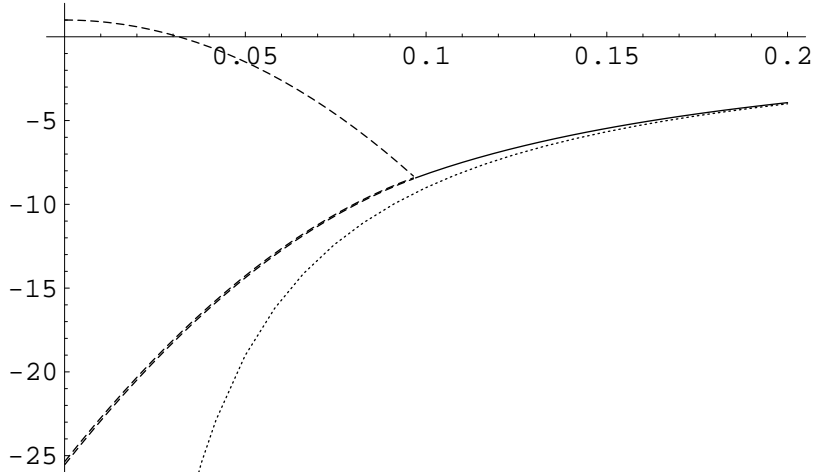


Figure 1: *The solid line shows the numerical solution obtained for the Schwarzschild function $A(r)$ for $r > r_0 = 0.1$. The possible matching solutions for $r < r_0$ are plotted in wide dashes. The upper curve represents an identically vanishing $D(r)$, the two, very similar, lower curves are obtained, respectively, from a linear $D(r)$ and from numerical integration, as explained in the text. The curve in close dashes presents the standard Schwarzschild solution $A(r) = 1 - \frac{2m}{r}$ of general relativity with its horizon fixed at the value $r = 1$.*

series does, which has also been confirmed numerically. Hence, the singularity of the negative energy solution does not get regularized and marks this solution as unphysical. This is a remarkable example of how a gravity theory improves the singularity situation by getting rid of the annoying ones while keeping those which are important for a stable Minkowski vacuum.

Some other points are interesting. The positive energy solutions of figure 1 which just regularize the metric function do only have a single horizon. The solution regularizing also the curvature singularities has two horizons. This confirms and exemplifies an argument of [14], which says that the number of horizons in a solution of higher derivative gravity that regularizes the curvature singularities at $r = 0$ has to be even.

Figure 1 has been plotted for $2m = 1$ and $\lambda = 10^{-3}$, i.e., for the ratio $2m/\lambda \gg 1$. The solutions undergo a qualitative change for smaller masses. To see this, figure 2 shows the numerical solutions for the same $2m\lambda$ but for different ratios $2m/\lambda$. For masses that are smaller than λ , the horizons disappear and the Schwarzschild function stays positive for all values of r . This phenomenon has been christened a ‘bare mass’ in [6]. The existence of

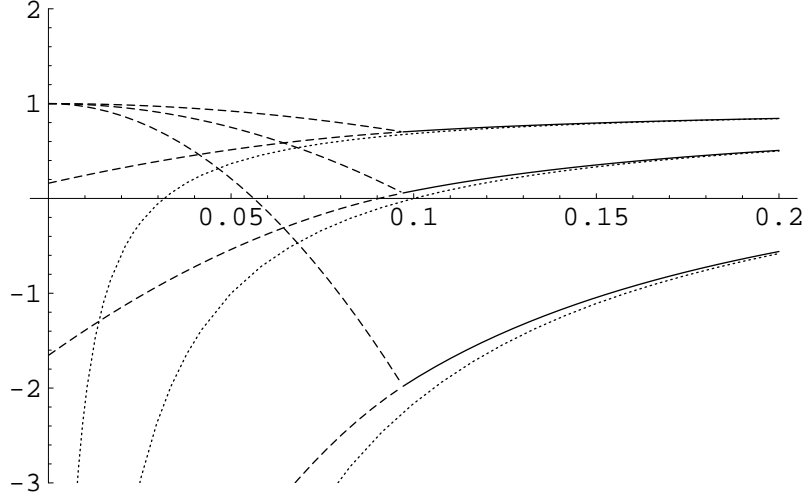


Figure 2: *The solutions of the Schwarzschild function $A(r)$ for fixed $2m\lambda = 10^{-3}$, but different ratios $2m/\lambda = 10^2, 10, 1$ from bottom to top. The closely dashed lines show the corresponding Schwarzschild solutions of general relativity with their respective horizons at $10^{-1/2}, 10^{-1}, 10^{-3/2}$.*

such small bare masses, however, is unclear. As will be seen below, in string theory, the parameter λ would naturally be of the order of magnitude of α' . On the other hand, it is a nice feature of a gravity theory that extremely small, but also extremely localized, energy fluctuations do only produce a small disturbance of Minkowski spacetime without a horizon.

III. CANCELLATION OF GHOSTS

Ghost freedom is a necessary requirement for any theory of gravity to be of possible relevance in a fundamental unified quantum theory. To achieve ghost freedom in gravity *à la* Born-Infeld, one has to go beyond the basic model presented in the preceding section.

While keeping the nice determinantal form of the action, it is not sufficient to consider only terms of first order in the curvature inside the determinant; second order terms have to be included as well. The most general action, obeying the construction principle of antisymmetrized indices used above, is of the form

$$\int \sqrt{-g} \left[(\det(\delta_B^A + \lambda M_B^A + \lambda^2 N_B^A))^{\zeta} - 1 \right] \quad (17)$$

where M_B^A and N_B^A contain all possible first and second order terms, respectively. Hence, a curvature expansion of the action directly corresponds to an expansion in orders of the

parameter λ . In general,

$$M_B^A = \kappa R \delta_B^A + R_B^A + \mu S_B^A, \quad (18a)$$

$$\begin{aligned} N_B^A = & (\rho R^2 + \sigma R_{ab}^2 + \alpha R_{abcd}^2) \delta_B^A + \beta R R_B^A + \tau R S_B^A + \gamma R_C^A R_B^C \\ & + \delta_1 R_C^A S_B^C + \delta_2 S_C^A R_B^C + \xi S_C^A S_B^C + \omega_1 R_{1B}^A + \omega_2 R_{2B}^A + \omega_3 R_{3B}^A \end{aligned} \quad (18b)$$

where fourteen new parameters occur which are denoted by greek characters. The tensors S and R_i are defined in the appendix, and some of their properties are given. Here, it suffices to say that S essentially contains the Ricci tensor, and the R_i present different contractions of two Riemann tensors.

Substituting the above expressions for M and N into the curvature expansion formula (A9), one finds the following linear and quadratic terms,

$$\frac{1}{2} \lambda f_0 R + \frac{1}{2} \lambda^2 (f_1 R^2 - 4f_2 R_{ab}^2 + f_3 R_{abcd}^2), \quad (19)$$

where some functions f_i of the parameters in the model appear. Ghost freedom is implied by the Gauss-Bonnet combination (2) of the quadratic curvature terms which in turn follows from the two equations $f_1 = f_2 = f_3$. More specifically,

$$f_0 = 1 + (d-1)(d\kappa + 2\mu), \quad (20a)$$

$$f_1 = \frac{\zeta}{4} f_0^2 - \frac{d-1}{2} (d\kappa^2 + 4\kappa\mu - 2d\rho - 4\tau) - \kappa - \mu^2 + \beta + 2\xi, \quad (20b)$$

$$f_2 = -\frac{d(d-1)}{4} \sigma - \frac{1}{2} (\delta_1 + \delta_2 - \mu + \omega_3) + \frac{d-2}{4} (\mu^2 - 2\xi), \quad (20c)$$

$$f_3 = d(d-1)\alpha + \frac{1}{2} \gamma - \frac{1}{4} + \omega_1 + 2(d-1)\omega_2 - \omega_3. \quad (20d)$$

This implies that there is a huge class of ghost-free models; this class is a thirteen-parameter family because the fourteen new parameters are only constrained by the following two equations,

$$\begin{aligned} & d(d-1)(\sigma + 4\alpha) + 2(\gamma + \delta_1 + \delta_2) - (d-2)(\mu^2 - 2\xi) \\ & - 2\mu + 4\omega_1 + 8(d-1)\omega_2 - 2\omega_3 - 1 = 0, \end{aligned} \quad (21a)$$

$$\begin{aligned} & \zeta (1 + (d-1)(d\kappa + 2\mu))^2 - (d-1)(2d\kappa^2 + 8\kappa\mu - 4d\rho - d\sigma - 8\tau) \\ & + 2(\delta_1 + \delta_2 - \mu + \omega_3 - 2\kappa + 2\beta) - (d+2)(\mu^2 - 2\xi) = 0, \end{aligned} \quad (21b)$$

and the expansion parameter λ is free.

Unfortunately, the equations of motion for these models, even in the simple case of a spherically symmetric spacetime, are extremely involved. It is expected that, at least in some cases, the nice features, and essentially the regularization property, of the basic model of the preceding section are kept. This has been checked, for instance, for the specific parameter values $\lambda \neq 0$, $\beta = -\frac{1}{24}$ and $\gamma = \frac{1}{2}$ (with all remaining parameters vanishing). All properties were found to be qualitatively similar.

Clearly, if the gravity theories in Born-Infeld form were expected to play an important role in any fundamental theory, one would need further physical principles to narrow down the wide range of possible choices. One such, very constraining, principle might be the requirement of supersymmetrizability of the gravitational action. Another might be the idea that such an action could be derived as an effective action from string theory, as it has been possible in the case of Born-Infeld electrodynamics.

IV. CORRESPONDENCE TO STRING THEORY

Two different methods have been used to deduce the Born-Infeld action from string theory. One uses the Polyakov path integral [2], in which the open string degrees of freedom are integrated out, in order to derive the effective action for the appropriate background fields. The other method requires conformal invariance on the string worldsheet and constructs the corresponding beta functions [15]. The conditions that they vanish can be obtained as equations of motion from the effective action. The first approach is not available in the gravitational case. The Polyakov path integral is the connected generating functional and contains the effect of massless poles, and so it should not have a meaningful low energy expansion [16]. The approach via the gravitational beta functions is a valid one. This has been shown, for example, in [17] for general string gravity backgrounds, including the antisymmetric tensor field and the dilaton. But so far, this method has not been as successful as in the Born-Infeld case. The effective gravity action derived from string theory is only known for some low orders in the curvature.

Nevertheless, the known first few orders of effective gravity deduced from string theory can be used to narrow down the choice of parameters for the ghost-free models of the preceding section. This is done under the working hypothesis that such an action might indeed be of importance in a fundamental theory. A ghost-free model has to be taken because string

induced gravity is ghost-free [18]. The effective action employed here contains curvature terms up to the third order [19] and reads

$$\int \sqrt{-g} \left[R + \alpha' \text{Tr}(R \cdot R) + \alpha'^2 \left(\frac{1}{2} \text{Tr}(R \cdot R \cdot R) - \frac{1}{12} \text{Tr}(R \cdot R_3) + \frac{3}{2} \text{Tr}(S \cdot R \cdot R) - \frac{3}{8} R \text{Tr}(R \cdot R) \right) \right] \quad (22)$$

where the notation introduced in the appendix has been used. For illustrational purposes, and because the Born-Infeld style gravities at present are purely bosonic theories, the action for a bosonic σ -model is used in this section. In the supersymmetric case, the situation presents itself in a different light. Then the σ -models do not lead to third order curvature corrections of the Einstein action [20]. Terms of fourth order, however, are important [21] and have been paid a lot of attention. This is also due to the fact that they are the first non-trivial, supersymmetrizable (see, e.g., [22, 23]) corrections arising from string theories which are compactified to four dimensions. The second order terms appear in the Gauss-Bonnet combination, while the third order ones are not supersymmetrizable.

Note that the above action does seem to contain ghosts. This is because string scattering amplitudes, and beta functions, are calculated for external gravitons that are on Einstein shell, i.e., that satisfy the (linearized) vacuum Einstein equations $R_{\mu\nu} = 0$. This means that, in second order of the curvature, the terms R^2 and R_{ab}^2 simply can be added with the appropriate factors to achieve the Gauss-Bonnet combination [24] (the same can be effected by means of appropriate field redefinitions [17]). More generally, it means that a string-induced effective gravity action will never contain any terms that involve at least twice the Ricci tensor or Ricci scalar. Upon variation, one of them will always remain so that these terms vanish on shell, compare [25]. So the first allowed simplification in the Born-Infeld gravity models, which does not interfere with their consistency with string theory, is to set $\rho = \sigma = \tau = \xi = 0$; compare equation (18).

In addition to the two equations (21) that enforce Gauss-Bonnet on the quadratic level, there are further equations that constrain the remaining parameters. The first comes from comparing the first two orders of the curvature, see (19ff), and the string-induced action,

$$\alpha' f_0 = 4\lambda f_3. \quad (23)$$

Four equations are derived from comparing the third order terms in the curvature expansion. For this purpose, substitute (18) into (A9) and utilize the trace relations (A2)-(A5), which

leads to

$$\alpha'^2 f_0 = -\frac{4}{3}\lambda^2 (3\gamma - 1), \quad (24a)$$

$$\alpha'^2 f_0 = 24\lambda^2 \omega_3, \quad (24b)$$

$$\alpha'^2 f_0 = -\frac{4}{3}\lambda^2 [\gamma\mu - \mu + \delta_1 + \delta_2 + 2\omega_2 + 2\mu(\omega_1 + (d-2)\omega_2 - \omega_3)], \quad (24c)$$

$$\alpha'^2 f_0 = -\frac{4}{3}\lambda^2 [f_0 \{4\zeta(\omega_1 + 2(d-1)\omega_2 - \omega_3 + d(d-1)\alpha) - 8\alpha + \zeta(2\gamma - 1)\} \\ - 4\{\kappa(\gamma - 1) + \beta + 2\kappa(\omega_1 + 2(d-1)\omega_2 - \omega_3) + 4\mu\omega_2\}]. \quad (24d)$$

A last equation in the comparison at third order arises from the combination $\text{Tr}(R \cdot R_1)$ that is not present in the string-induced action. It appears, however, in the curvature expansion of the Born-Infeld style models, and so it has to cancel. This simply gives the condition

$$\omega_1 = 0. \quad (25)$$

So all extended Born-Infeld gravity models of the kind defined in section III in (17ff) whose parameters satisfy the constraint equations (21) and (23)-(24) are consistent with string-induced effective gravity at least up to third order of the curvature. In principle, this check could be continued to higher orders with the aim of finding further consistency conditions (or possibly an inconsistency with the theoretical σ -model predictions). This is, however, not a very interesting task. More important is the following observation.

From the constraint equations, it is clear that the expansion parameter λ is proportional to the inverse string tension, $\lambda \sim \alpha'$. This, again, is very reminiscent of Born-Infeld electrodynamics, where the action takes into account all orders in α' and, hence, all orders of the field strength tensor, which is responsible for the regularization of the electric field of point charges. The fascinating idea here is the suggestion of a very similar mechanism for gravity. By taking into account all orders of α' in curvature corrections in the string-induced gravity action, string theory might resolve the singularity problems of general relativity. The theories presented here are the first examples of gravity theories doing just that. Whether or not one of them is indeed derivable from string/M-theory cannot be answered at present.

V. DISCUSSION

A simple recipe has been presented for constructing gravity theories with an action in determinantal form, very similar to the action of Born-Infeld electrodynamics. The construc-

tion is based on the observation that the Schwarzschild singularity can only be recognized from the second order curvature invariant R_{abcd}^2 , but not from those invariants constructed from the Ricci tensor. Hence, a gravity theory that expects to cure singularities, by placing upper bounds on R_{abcd}^2 , or otherwise, should include the full Riemann tensor. The actual formulation of the new class of theories presented in this article utilizes the symmetries of the Riemann tensor, considering only tensors with pairs of antisymmetrized indices.

For the simplest version of Born-Infeld style gravity, the analogue of the spherically symmetric Schwarzschild solution has been analyzed. A different behaviour has been found in different mass regimes. In all cases the Schwarzschild singularity, at least for the positive energy solution, may be removed (a detailed discussion is found in section II B). For large masses, compared to some expansion parameter, the horizon remains; small masses, on the other hand, become ‘bare masses’ without a horizon. The negative energy Schwarzschild solution remains singular and is thus distinguished to be unphysical, which is necessary in order to have a stable Minkowski vacuum configuration. The bare masses are a nice feature in the sense that extremely small yet localized energy fluctuations would only produce small spacetime distortions, without generating horizons and thereby changing the topology. These characteristics make the new Born-Infeld style gravities nice examples of theories, in which the annoying singularities are removed, whereas the important ones remain.

It has been shown that extended models can be defined in which the ghosts, in a perturbative expansion around flat Minkowski spacetime, cancel. The improved behaviour of these theories with respect to general relativity is kept, at least for some models. Unfortunately, the calculations, even for the simple spherically symmetric spacetime, are quite involved, so that it remains unclear whether all extended, ghost-free models have improved singularity properties.

Ghost-freedom is essential, if the Born-Infeld gravities were expected to be derivable from an underlying, unified theory such as string or M-theory. The consistency with string-induced gravity, has been illustrated for a bosonic σ -model, at least up to third order in the curvature. In checking this correspondence, all influence of the dilaton field and of the antisymmetric tensor field of the string gravity backgrounds has been neglected. It might be of interest to pursue extensions of the Born-Infeld style gravities which try to incorporate these fields. The value of the string theory against gravity comparison is not so much a proof, showing exactly what kind of gravity is induced by string theory, but much more the

exhibition of the following interesting and important mechanism.

A way has been suggested in which string theory, by inclusion of all orders of α' corrections, corresponding to all orders of the curvature in the string-induced gravity action, might actually regularize gravitational singularities. The final gravity theory induced from string or M-theory thus is expected to have at least all the good features of the gravities *à la* Born-Infeld, which are nice examples realizing this idea. Furthermore, it would probably be supersymmetrizable, and it would be interesting to investigate whether any of the models here has this property.

Appendix A: SOME TECHNICALITIES

This appendix contains some calculations and definitions needed in the main text, especially in section III, which considers the ghost-free models of Born-Infeld gravity, and in section IV, where these models are compared to string-induced effective gravity.

The following tensors are needed for the generalized models. They contain the Ricci tensor at the first order level (which cannot be obtained by tracing R^A_B) and some special combinations of two Riemann tensors in second order of the curvature:

$$S^A_B \equiv R^{a_1}_{b_1} \delta^{a_2}_{b_2} - R^{a_2}_{b_1} \delta^{a_1}_{b_2} - (b_1 \leftrightarrow b_2), \quad (\text{A1a})$$

$$R^A_{1B} \equiv R^{a_1 a_2 cd} (R_{cb_1 db_2} - R_{cb_2 db_1}), \quad (\text{A1b})$$

$$R^A_{2B} \equiv R^{a_1 cde} R_{b_1 cde} \delta^{a_2}_{b_2} - R^{a_2 cde} R_{b_1 cde} \delta^{a_1}_{b_2} - (b_1 \leftrightarrow b_2), \quad (\text{A1c})$$

$$R^A_{3B} \equiv R^{a_1 cd}_{b_1} R^{a_2}_{cd b_2} - R^{a_2 cd}_{b_1} R^{a_1}_{cd b_2} - (b_1 \leftrightarrow b_2). \quad (\text{A1d})$$

These tensors and their products, including also the Riemann tensor R^A_B , obey the following important trace relations, in which the notation $(R \cdot S)^A_B = R^A_C S^C_B$ is used,

$$\text{Tr } \delta = \frac{d(d-1)}{2}, \quad \text{Tr } R = \frac{1}{2} R, \quad \text{Tr } S = (d-1) R, \quad (\text{A2a})$$

$$\text{Tr } R_1 = \frac{1}{2} R^2_{abcd}, \quad \text{Tr } R_2 = (d-1) R^2_{abcd}, \quad \text{Tr } R_3 = R^2_{ab} - \frac{1}{2} R^2_{abcd}, \quad (\text{A2b})$$

and

$$\text{Tr}(R \cdot R) = \frac{1}{4} R_{abcd}^2, \quad (\text{A3a})$$

$$\text{Tr}(R \cdot S) = R_{ab}^2, \quad (\text{A3b})$$

$$\text{Tr}(R \cdot R_1) = \frac{1}{2} R^{abcd} R_{cdef} R_{ab}^{ef}, \quad (\text{A3c})$$

$$\text{Tr}(R \cdot R_2) = 2\text{Tr}(SRR), \quad (\text{A3d})$$

$$\text{Tr}(R \cdot R_3) = R^{abcd} R_{cefa} R_d{}^{ef}{}_b, \quad (\text{A3e})$$

as well as

$$\text{Tr}(S \cdot S) = R^2 + (d-2)R_{ab}^2, \quad (\text{A4a})$$

$$\text{Tr}(S \cdot R_1) = 2\text{Tr}(S \cdot R \cdot R), \quad (\text{A4b})$$

$$\text{Tr}(S \cdot R_2) = 2(d-2)\text{Tr}(S \cdot R \cdot R) + 4R\text{Tr}(R \cdot R), \quad (\text{A4c})$$

$$\text{Tr}(S \cdot R_3) = -2R_{ab}R_{cd}R^{acdb} - 2\text{Tr}(S \cdot R \cdot R), \quad (\text{A4d})$$

and

$$\text{Tr}(R \cdot R \cdot R) = \frac{1}{8} R^{abcd} R_{cdef} R_{ab}^{ef}, \quad (\text{A5a})$$

$$\text{Tr}(S \cdot R \cdot R) = R_{ab} R^{acde} R_{cde}^b. \quad (\text{A5b})$$

In all calculations involving tensors with antisymmetrized index pairs, care has to be taken that a sum over capital indices is equivalent to a sum over unordered normal indices multiplied by a factor of one half, i.e., symbolically

$$\sum_A \longleftrightarrow \frac{1}{2} \sum_{a_1, a_2}. \quad (\text{A6})$$

The trace identities are necessary in order to evaluate the curvature expansion of the Lagrangian, i.e., of the function

$$(\det(\delta_B^A + \lambda M_B^A + \lambda^2 N_B^A))^\zeta \quad (\text{A7})$$

where M and N are of first or second order in the curvature, respectively. This curvature expansion is equivalent to an expansion around $\lambda = 0$ and is obtained by using the matrix formula

$$\det(\delta + C) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{Tr}(C^n) \right)^m \quad (\text{A8})$$

which follows from the identity $\ln \det(\delta + C) = \text{Tr} \ln(\delta + C)$ and the series expansions of the logarithm and the exponential. Up to third order, the result is

$$1 + \zeta \left[\lambda \text{Tr} M + \lambda^2/2 (2\text{Tr} N + \zeta(\text{Tr} M)^2 - \text{Tr}(M^2)) + \lambda^3/6 (\zeta^2(\text{Tr} M)^3 + 6\zeta \text{Tr} M \text{Tr} N - 3\zeta \text{Tr} M \text{Tr}(M^2) - 6\text{Tr}(MN) + 2\text{Tr}(M^3)) + \mathcal{O}(\lambda^4) \right]. \quad (\text{A9})$$

Appendix B: CORRIGENDUM

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In the paper above, a method for the construction of novel gravity actions of Born-Infeld type is presented. (Complete information on the Riemann tensor is encoded and could enable improved singularity behaviour, which is embellished by relations to theories with tidal acceleration bounds [26].)

A few points need correction and clarification. Expression (11) for the effective action of the basic model on static spherically symmetric spacetimes is incorrect; it should read

$$\int dr \sqrt{AB} r^2 \left[\frac{1}{2r} A^{-2/3} B^{-3/2} \left((\lambda + B(r^2 - \lambda))(2rAB + \lambda A')^2 (2rB^2 - \lambda B')^2 \times \right. \right. \\ \left. \left. \times (4A^2 B^2 - \lambda(AA'B' + A'^2 B - 2AA''B)) \right)^{1/6} - 1 \right]. \quad (\text{B11})$$

This correction affects the term -1 within the square brackets of the original equation in the paper which is replaced by $-(AB)^{-1/2}r$ here. The curvature invariants on four-dimensional de Sitter space, given below figure 1, should read $R = 12/\lambda$, $R_{ab}^2 = 36/\lambda^2$ and $R_{abcd}^2 = 24/\lambda^2$, in the usual convention that the Riemann tensor of a space of constant curvature $1/\lambda$ is $R_{abcd} = (g_{ac}g_{bd} - g_{ad}g_{bc})/\lambda$. The first misprint was pointed out by Deser *et al.* [27]; subsequent computations are unaffected.

Another important point has been raised by the authors of [27]. Given any physical theory in terms of its action, it is generally not the case that solutions to the equations of motion of an effective theory obtained by a truncation of the original action (by means of a specific ansatz for the relevant fields) are also solutions to the equations of motion of the original theory. In other words, truncations in this sense and variation generally do not commute. Nevertheless, it has been shown [8] that the effective action (11) for a static spherically symmetric spacetime ansatz with two undetermined functions A and B of the radial coordinate may be considered.

In the paper, a further truncation of the static spherically symmetric ansatz, setting $B = 1/A$, has been used in deriving the effective action (13) and the equation of motion (14). This truncation is not related to any symmetry; thus, equation (14) is necessary, but not sufficient: any solution of the equations for A and B from (11) which satisfies $B = 1/A$ solves (14), but the converse is not generally true. Therefore, the validity of any solution of (14) has to be checked by substituting it into at least one of the equations for A or B from (11). The procedure of using the simpler equation (14) should be considered merely a means of suggesting solutions for the original equations.

The paper gives $c_1 r^3$, $c_1 r^2$ and $c_1 r + c_2$ for arbitrary constants c_i as solutions of (14). Unfortunately, a check of the validity of these solutions in the equations derived from (11) was omitted, and we will do this check now. We find that $c_1 r^3$ is invalid. However, $c_1 r^2$ gives valid solutions, provided either $c_1 = -3$ or $c_1 = 1$, where the constant c_1 is determined as a real solution of the equation $2|c_1| = c_1 + c_1^2$ (which implies the equations from (11) with $B = 1/A$). For the linear expression $c_1 r + c_2$, a denominator vanishes in the equations derived from both actions (11) and (13), so that this expression cannot be a stationary solution.

It has not been properly checked yet whether the previously given numerical solutions of (14) are correct. Although a substitution of the solution into the equations from (11) produces ‘small’ errors, and although the error associated with the equation of motion for A oscillates around zero, such a check is not conclusive in many examples of differential equations. A proper check would involve the integration of both equations for A and B .

ACKNOWLEDGMENTS

It is a pleasure to thank Paul K. Townsend for helpful discussions, and Bob Holdom, Donald Marolf and Frederic P. Schuller for useful e-mails. I also wish to thank Stanley Deser, Bayram Tekin and Joel Franklin for e-mail correspondence, drawing to my attention refs. [8, 9] on symmetry reductions and pointing out some mistakes. Financial support from the Gates Cambridge Trust is gratefully acknowledged.

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